代数幾何学シンポジウム記録
2009年度　pp.112-112

Resolution of dihedral orbifolds

Alvaro Nolla de Celis, alnolla@gmail.com
(Nagoya University)

Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be the following small binary dihedral group:

$$G = \left\langle \alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix},\ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} :\ \varepsilon^{2n} = 1,\ (2n,a) = 1,\ a^2 \equiv 1 \pmod{2n} \right\rangle$$

where $A = \langle \alpha \rangle = \frac{1}{2n}(1,a)$ is a maximal normal of index 2, and we consider the minimal resolution $Y \to \mathbb{C}^2/G$.

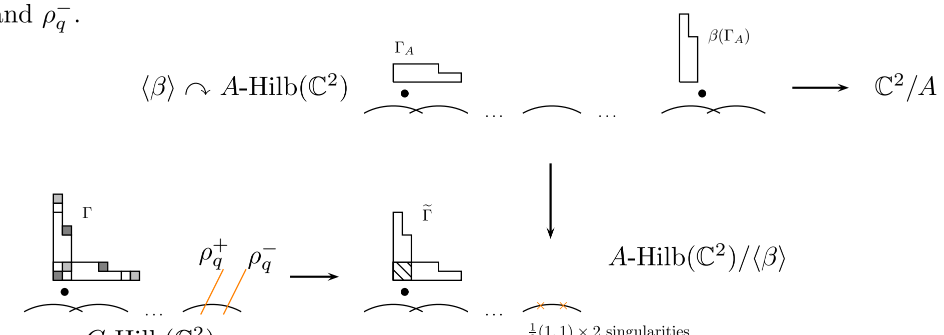
$$\begin{aligned} Y &= G\text{-Hilb}(\mathbb{C}^2) : \text{Moduli space parametrising G-clusters ([Ishii])} \\ &= \mathcal{M}_\theta(Q,R) : \text{Moduli of } \theta\text{-stable representations of the bound McKay quiver} \end{aligned}$$

Definition: Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite subgroup. A *G-graph* is a subset $\Gamma \subset \mathbb{C}[x,y]$ such that it contains $\dim\rho$ elements in each irreducible representation ρ.

Motivation: For any G-cluster $\mathcal{Z} \in G\text{-Hilb}(\mathbb{C}^2)$, the basis of $\mathcal{O}_\mathcal{Z}$ as a vector space is a G-graph. Given a G-graph Γ, all the G-clusters with Γ as basis for $\mathcal{O}_\mathcal{Z}$ form an open set $U_\Gamma \subset G\text{-Hilb}(\mathbb{C}^2)$, and the collection of distinguished $\{U_{\Gamma_i}\}_{i\in I}$ covers $G\text{-Hilb}(\mathbb{C}^2)$.

General construction

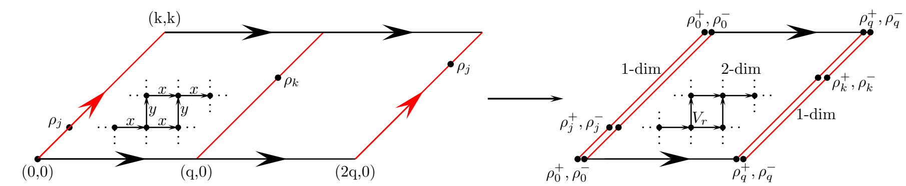
- *Fact:* $G\text{-Hilb}(\mathbb{C}^2) = G/A\text{-Hilb}(A\text{-Hilb}(\mathbb{C}^2))$.
- The symmetry of the continued fraction $\frac{2n}{a}$ implies that (i) the coordinates along the exceptional divisor $E \subset A\text{-Hilb}(\mathbb{C}^2)$ are also symmetric, and (ii) β is an involution on the middle curve $E_m \cong \mathbb{P}^1$ on the exceptional divisor on $A\text{-Hilb}(\mathbb{C}^2)$.
- Every G-graph Γ is either the unique extension of the union of two symmetric A-graphs, or it comes from a choice on the special irreducible representations ρ_q^+ and ρ_q^-.



Orbifold McKay quiver

Let $\mathrm{Irr}\,A = \{\rho_0, \ldots, \rho_{2n-1}\}$ the irreducible representations of A. The McKay quiver of $A = \frac{1}{2n}(1,a)$ can be written on a torus, and the quotient $G/A \cong \mathbb{Z}/2 = \langle\beta\rangle$ acts on $\mathrm{Irr}\,A$ by conjugation. Then

McKay quiver of G = $\mathbb{Z}/2$-orbifold of the McKay quiver of A



Fixed points $\rho_j \in \mathrm{Irr}\,A$ by β become two 1-dimensional representations ρ_j^+ and ρ_j^-. Free orbits $\{\rho_r, \rho_{ar}\}$ by β become one 2-dimensional representation V_r.

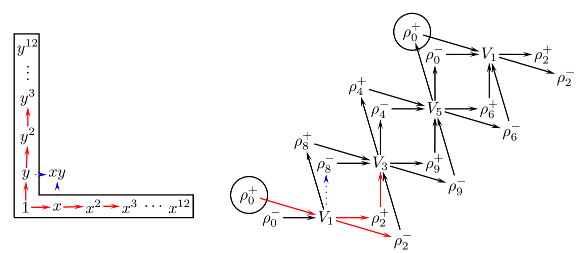
Explicit description of a open cover of Y

Let (Q,R) the bound McKay quiver, $\mathbf{d} = (\dim\rho_i)_{i\in Q_0}$ the dimension vector and the generic stability condition $\theta = (-\sum_{\rho_i \in \mathrm{Irr}\,G} \dim\rho_i, 1\ldots, 1)$. This choice of θ implies that *there exist* $\dim\rho_j$ *nonzero paths from the distinguished source* ρ_0^+ *to every other irreducible representation* ρ_j.

Any G-graph Γ produces the choices for nonzero maps in the representation space of (Q,R). Therefore, given any G-graph Γ we can associate an open set $M_\Gamma \subset \mathcal{M}_\theta(Q,R)$, and the $\{M_{\Gamma_i}\}_{i\in I}$ covers $\mathcal{M}_\theta(Q,R)$.

Using the relations R of Q the equations of M_Γ are explicitly obtained.

Example: Let $G = \langle \frac{1}{12}(1,7),\ \beta \rangle$. The minimal resolution Y consists of 5 open sets given by the G-graphs $\Gamma_2, \ldots, \Gamma_5$. For instance, for the G-graph Γ_0 we have:



Remaining open sets for $\mathcal{M}_\theta(Q,R)$ as hypersurfaces in $\mathbb{C}^3$:

$$\begin{aligned} M_{\Gamma_2} &: \ b_2^+ E = (b_2^+ + 1) D^+ \\ M_{\Gamma_3} &: \ b_2^- G = (b_2^- + 1) D^- \\ M_{\Gamma_4} &: \ ef = (e^2 f - 1) D_+ \\ M_{\Gamma_5} &: \ gh = (g^2 h - 1) D_- \end{aligned}$$

and M_{Γ_1} is given by $(cd = (1 + cd^2)E) \subset \mathbb{C}^3$